Generalized Formulation of Nonlinear Pitch-Yaw-Roll Coupling: Part II—Nonlinear Coning-Rate Dependence

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The aerodynamic formulation of Part I is further generalized to eliminate the assumption of a linear dependence of the moment on coning rate. Two of the four previously described motions combine, so that the total moment is compounded of the contributions from three simple motions. The basic motion is coning, where the nose of the aircraft describes a circle around the velocity vector, while the remaining motions are oscillatory perturbations carried out in the presence of coning. A re-examination of the assumptions underlying the formulation enables a characterization of aerodynamic phenomena whose effects can and cannot be treated within the scope of the formulation. Recommendations are made as to the most appropriate types of wind-tunnel tests that could be undertaken in fulfillment of the formulation's requirements.

Introduction

TWO significant studies[1,2] encourage the belief that the results of wind-tunnel tests can be used successfully to predict flight behavior at large angles of attack. They differ, however, in the adoption of an aerodynamic formulation, and therefore, in the type of testing advocated. In Ref. 1, the actual spins of an F-100 aircraft were reproduced by calculations based on an aerodynamic formulation that called for principally wind-tunnel measurements of the conventional static forces and moments. In Ref. 2, the spins of a delta-wing aircraft model in a spin tunnel were reproduced by calculations based on an aerodynamic formulation that called for principally wind-tunnel measurements of the forces and moments on a model in coning motion. While both studies achieved a considerable measure of success, in neither study was any particular attention devoted to an analysis or justification of the aerodynamic formulation that was adopted, although this determined the form of testing undertaken and advocated as successful.

There still exists a need for an aerodynamic formulation applicable to high-angle-of-attack maneuvers and defensible on theoretical rather than pragmatic grounds, on the basis of which the most appropriate types of wind-tunnel tests could be selected. In Part I, the authors attempted to meet this need by proposing a nonlinear formulation of the aerodynamic moment system based on concepts from functional analysis. The total moment for an arbitrary motion was shown to be compounded of the contributions from four simple motions, one of which was the conventional steady motion proposed in Ref. 1 and another of which was the coning motion proposed in Ref. 2. However, the analysis in Part I contains an assumption that the moment contribution due to coning will depend only linearly on the coning rate. Physical reasoning suggests, and the experimental results in Ref. 2 confirm, that for motions within the stall and post-stall regimes the moment may show a significant and perhaps even critical nonlinear dependence on the coning rate. The formulation in Part I would apply more fully to motions in the stall and post-stall regimes if the assumption of a linear dependence on coning rate could be eliminated. This can in fact be done, and the primary purpose of Part II of this report will be to present the resulting less restrictive aerodynamic formulation. In addition, the fundamental assumptions on which the formulation rests will be re-examined with a view toward characterizing aerodynamic phenomena whose effects on motions can and cannot be treated within the scope of the extended formulation. Finally, recommendations will be made as to the most appropriate types of wind-tunnel testing that could be undertaken in fulfillment of the formulation's requirements.

Presented as Paper 74-85 at the AIAA 12th Aerospace Sciences Meeting, Washington, D.C., January 30–February 1, 1974; submitted June 13, 1974.

Index categories: Aircraft Handling, Stability, and Control; Aircraft Testing (Including Component Wind Tunnel Testing); Nonsteady Aerodynamics.

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Analysis

Nonlinear Formulation of Aerodynamic Moment System

It will be shown how the analysis contained in Part I can be recast to yield a formulation allowing an arbitrary nonlinear dependence on coning rate. The symbols, conventions, and coordinate systems used in Part I are retained in the following, as are the assumptions underlying the use and applicability of functional analysis. A later section will include a re-examination of the basic assumptions.

Approximate formulation in the aerodynamic axis system

In Part I, the components of the moment coefficient in the aerodynamic axis system were said to be functionals of the five argument functions δ, ψ, λ, q, r. For example, the pitching-moment coefficient C_m was specified to be a functional of the form

$$C_m(t) = G[\delta(\xi), \psi(\xi), \lambda(\xi), q(\xi), r(\xi)] \tag{1}$$

Simplification of the aerodynamic formulation based on Eq. (1) reduced the formulation to a sum of four terms, each representing the moment contribution due to a characteristic motion. One of these, the contribution due to coning motion of the longitudinal axis around the flight velocity vector, resulted from the combination of contributions due to separate motions in λ and r. Thus, on the basis of Eq. (1) as the functional representation, coning motion emerged as one of the characteristic motions as a result of the assumption of a linear dependence on the rates λ and r, which subsequently allowed the superposition of their separate moment contributions.

It is desired to retain coning motion as one of the characteristic motions in view of its close resemblance by itself to the classical spin motion. At the same time, its appearance is desired without the necessity of invoking the linearity assumption which was required in Part I in order to combine contributions. These ends can be accomplished by reconsidering Eq. (1). The argument functions λ, q, r in Eq. (1) comprise an orthogonal set and thus may be considered to be the components of a vector. Any other orthogonal set of functions comprising the same vector may be used as argument functions as well. In particular, let $\dot{\lambda}$, q, r be resolved into a set ω_1, ω_2, ω_3 by a rotation around y through the angle σ, so that ω_1 is directed along the flight velocity vector. Then

$$\left.\begin{aligned} \omega_1 &= \dot{\lambda}\cos\sigma + r\sin\sigma = \gamma\dot{\lambda} + \delta r \\ \omega_2 &= q \\ \omega_3 &= -\dot{\lambda}\sin\sigma + r\cos\sigma = -\delta\dot{\lambda} + \gamma r \end{aligned}\right\} \tag{2}$$

The functional representation of $C_m(t)$ becomes

$$C_m(t) = G_1[\delta(\xi), \psi(\xi), \omega_1(\xi), \omega_2(\xi), \omega_3(\xi)] \tag{3}$$

On the basis of this representation, new indicial functions and a new integral form for $C_m(t)$ may be formulated in the same way as before. Expanding the integral form as before, but now assuming that only ω_2 and ω_3 are small, yields to first order in ω_2 and ω_3

$$\begin{aligned} C_m(t) = {} & C_m(\infty\,; \delta(t), \psi(t), \omega_1(t)) + \\ & \frac{\omega_2 l}{V} C_{m_{\omega_2}}(\infty\,; \delta(t), \psi(t), \omega_1(t)) + \\ & \frac{\omega_3 l}{V} C_{m_{\omega_3}}(\infty\,; \delta(t), \psi(t), \omega_1(t)) + \\ & \frac{\dot{\delta} l}{V} C_{m_{\dot{\delta}}}(\delta(t), \psi(t), \omega_1(t)) + \frac{\dot{\psi} l}{V} C_{m_{\dot{\psi}}}(\delta(t), \psi(t), \omega_1(t)) \end{aligned} \tag{4}$$

where, in the functional dependence notation, the infinity symbol indicates steady flow, and the zeros belonging to ω_2 and ω_3 have been omitted. A term multiplied by $\dot{\omega}_1$ resulting from the expansion of the integral also has been omitted on the assumption that while $C_m(t)$ may show a significant nonlinear dependence on ω_1, even for small ω_1, the dependence on $\dot{\omega}_1$ is of second order in frequency and hence negligible for small frequencies.

Since the terms in Eq. (4) depend on ω_1, which is itself the coning rate of the longitudinal axis around the flight velocity vector, coning motion will now be a characteristic motion without the necessity of additional operations. Further, the dependence of the moment coefficients on coning rate is now arbitrary; the restriction to a linear dependence has been eliminated.

Simplification of the formulation in the aerodynamic axis system

As before, Eq. (4) may be simplified on the assumption of small plunging of the center of gravity, where terms multiplied by $q-\dot{\sigma}$ and $r-\varepsilon\dot{\lambda}$ will be negligibly small. It will be noted from Eq. (2) that $\omega_1 - \dot{\lambda}/\gamma = \delta(r-\varepsilon\dot{\lambda})$ and $\omega_3 = \gamma(r-\varepsilon\dot{\lambda})$. Hence, if the terms in Eq. (4) are expanded to first order around $\omega_1 = \dot{\lambda}/\gamma$, then terms multiplied by $\omega_1 - \dot{\lambda}/\gamma$ and ω_3 may be discarded on the basis of negligibly small $r-\varepsilon\dot{\lambda}$. Carrying out the expansion, adding and subtracting the term $(\dot{\sigma}l/V)C_{m_{\omega_2}}$, $(C_{m_{\omega_2}} \equiv C_{m_q})$ and rearranging, yields

$$\begin{aligned} C_m(t) = C_m\left(\infty\,; \delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t)\right) + \frac{\dot{\sigma} l}{V} C_{m_{\dot{\sigma}}}\left(\delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t)\right) + \\ \frac{\dot{\psi} l}{V} C_{m_{\dot{\psi}}}\left(\delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t)\right) \end{aligned} \tag{5}$$

where

$$C_{m_{\dot{\sigma}}} = C_{m_q} + \gamma C_{m_{\dot{\delta}}} \tag{6}$$

and terms multiplied by $q-\dot{\sigma}$ and $r-\varepsilon\dot{\lambda}$ have been omitted. The terms surviving in Eq. (5) are identified by comparing them with those obtained in the case of zero plunging where $q-\dot{\sigma}$ and $r-\varepsilon\dot{\lambda}$ are identically zero. The first term is the pitching-moment coefficient that would be measured in a steady coning motion δ = const, ψ = const, $\omega_1 = \dot{\lambda}/\gamma$ = const. As before, the term $C_{m_{\dot{\sigma}}}$ is the damping-in-pitch coefficient that would be measured from small oscillations in σ about σ = const with ψ fixed at ψ = const, but now, in addition, *in the presence of a steady coning motion* $\omega_1 = \dot{\lambda}/\gamma$ = const. Similarly, $C_{m_{\dot{\psi}}}$ is the damping-in-roll coefficient that would be measured from small oscillations in ψ about ψ = const with δ fixed at δ = const and in the presence of a steady coning motion $\omega_1 = \dot{\lambda}/\gamma$ = const. The indicated functional dependence on δ, ψ, $\dot{\lambda}/\gamma$ must be interpreted as follows: for flight with given values of δ, ψ, $\dot{\lambda}$, q, r at a particular instant, the aerodynamic coefficients that are to be associated with this instant are those evaluated around a coning motion having constant values of δ and ψ equal to the instantaneous flight values and a constant value of coning rate equal to the instantaneous flight value of $\dot{\lambda}/\gamma$.

Thus, the four moment contributions required in Part I to build up the response to an arbitrary motion reduce to three when a nonlinear dependence on coning rate is admitted. This is because the first term in Eq. (5), $C_m(\infty\,; \delta, \psi, \dot{\lambda}/\gamma)$, is the general term which, it now appears, replaces two terms in Part I representing the expansion of $C_m(\infty\,; \delta, \psi, \dot{\lambda}/\gamma)$ around $\dot{\lambda}/\gamma = 0$ to first order in $\dot{\lambda}/\gamma$. The more important change from Part I, at least from the experimental standpoint, is that retaining a nonlinear dependence on coning rate requires for consistency that the oscillatory experiments be carried out in the presence of coning motion.

In summary, with terms multiplied by $q-\dot{\sigma}$ and $r-\varepsilon\dot{\lambda}$ neglected, but with nonlinear dependence on coning rate included, the components of the aerodynamic moment system take the form

$$\begin{aligned} C_k(t) = C_k\left(\infty\,; \delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t)\right) + \frac{\dot{\sigma} l}{V} C_{k_{\dot{\sigma}}}\left(\delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t)\right) + \\ \frac{\dot{\psi} l}{V} C_{k_{\dot{\psi}}}\left(\delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t)\right) \qquad k = l, m, n \end{aligned} \tag{7}$$

In the aerodynamic axis system the three characteristic motions are coning, oscillations in pitch, and oscillations in roll, all evaluated at constant values of resultant angle of attack, roll angle, and coning rate. The motions are illustrated schematically in Fig. 1.

Approximate formulation in the body axis system and simplification

The analysis parallels that of the previous section. In Part I the functional representation of $\hat{C}_m(t)$ in body axis variables was specified as

$$\hat{C}_m(t) = H[\hat{\alpha}(\xi), \hat{\beta}(\xi), p_B(\xi), q_B(\xi), r_B(\xi)] \tag{8}$$

Here, the orthogonal set p_B, q_B, r_B is resolved into a set $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{\omega}_3$ by a rotation around x_B through the angle ψ and then around y through the angle σ, so that $\hat{\omega}_1$ is directed along the flight velocity vector. The new functional representation becomes

$$\hat{C}_m(t) = H_1[\hat{\alpha}(\xi), \hat{\beta}(\xi), \hat{\omega}_1(\xi), \hat{\omega}_2(\xi), \hat{\omega}_3(\xi)] \tag{9}$$

where

$$\begin{aligned} \hat{\omega}_1 &= p_B\cos\sigma + r\sin\sigma = \gamma p_B + \delta r \\ \hat{\omega}_2 &= q \\ \hat{\omega}_3 &= -p_B\sin\sigma + r\cos\sigma = -\delta p_B + \gamma r \end{aligned} \tag{10}$$

After forming indicial responses and an integral form for $\hat{C}_m(t)$ based on Eq. (9), expanding the integral form about $\hat{\omega}_2 = 0$, $\hat{\omega}_3 = 0$ yields to first order in $\hat{\omega}_2$ and $\hat{\omega}_3$

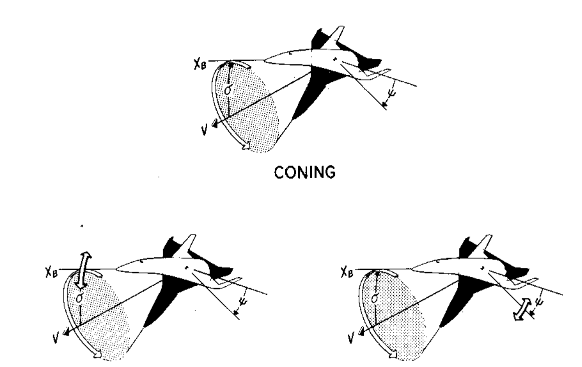

Fig. 1 Basic motions in aerodynamic axis system.

$$\hat{C}_m(t) = \hat{C}_m(\infty\,;\hat{\alpha}(t),\hat{\beta}(t),\hat{\omega}_1(t))+ \frac{\hat{\omega}_2 l}{V}\hat{C}_{m_{\hat{\omega}_2}}(\infty\,;\hat{\alpha}(t),\hat{\beta}(t),\hat{\omega}_1(t))+ \frac{\hat{\omega}_3 l}{V}\hat{C}_{m_{\hat{\omega}_3}}(\infty\,;\hat{\alpha}(t),\hat{\beta}(t),\hat{\omega}_1(t)) + \frac{\dot{\hat{\alpha}} l}{V}\hat{C}_{m_{\dot{\hat{\alpha}}}}(\hat{\alpha}(t),\hat{\beta}(t),\hat{\omega}_1(t))+ \frac{\dot{\hat{\beta}} l}{V}\hat{C}_{m_{\dot{\hat{\beta}}}}(\hat{\alpha}(t),\hat{\beta}(t),\hat{\omega}_1(t)) \qquad (11)$$

where the zeros belonging to $\hat{\omega}_2$ and $\hat{\omega}_3$ have been omitted. A term proportional to $\dot{\hat{\omega}}_1$ also has been omitted.

Expanding the terms in $\hat{C}_m(t)$ to first order around $\hat{\omega}_1 = p_B/\gamma$, invoking the small-plunging assumption $(q-\dot{\sigma}) \approx 0$, $(r-\varepsilon\dot{\lambda}) \approx 0$, and some manipulation allows rewriting Eq. (11) in a form analogous to that of Eq. (5) in the aerodynamic axis system:

$$\hat{C}_m(t) = \hat{C}_m\left(\infty\,;\hat{\alpha}(t),\hat{\beta}(t),\frac{p_B}{\gamma}(t)\right)+ \frac{1}{\gamma}\frac{\dot{\hat{\alpha}}l}{V}[\hat{C}_{m_{q_B}}(\infty\,;\ \hat{\alpha},\hat{\beta},p_B/\gamma)+\gamma\hat{C}_{m_{\dot{\hat{\alpha}}}}(\hat{\alpha},\hat{\beta},p_B/\gamma)] - \frac{1}{\gamma}\frac{\dot{\hat{\beta}}l}{V}[\hat{C}_{m_{r_B}}(\infty\,;\ \hat{\alpha},\hat{\beta},p_B/\gamma)-\gamma\hat{C}_{m_{\dot{\hat{\beta}}}}(\hat{\alpha},\hat{\beta},p_B/\gamma)] \qquad (12)$$

The first term is the pitching-moment coefficient along y_B that would be measured in a steady coning motion $\hat{\omega}_1 = p_B/\gamma = \text{const}$ with $\hat{\alpha}$ and $\hat{\beta}$ at the fixed inclinations $\hat{\alpha} = \text{const}$, $\hat{\beta} = \text{const}$. The second term is the damping-in-pitch coefficient that would be measured from small oscillations in $\hat{\alpha}$ about $\hat{\alpha} = \text{const}$ with $\hat{\beta}$ fixed at $\hat{\beta} = \text{const}$ and in the presence of a steady coning motion $\hat{\omega}_1 = p_B/\gamma = \text{const}$. The third term results from small oscillations in $\hat{\beta}$ about $\hat{\beta} = \text{const}$ with $\hat{\alpha}$ fixed at $\hat{\alpha} = \text{const}$ and in the presence of a steady coning motion $\hat{\omega}_1 = p_B/\gamma = \text{const}$. It should be noted that the coning rate $\hat{\omega}_1 = p_B/\gamma$ on which the terms depend in Eq. (12) is not equal in magnitude to the coning rate $\omega_1 = \dot{\lambda}/\gamma$ in Eq. (5). The rates differ by $\dot{\psi}/\gamma$. Thus, in the body axis system, the appropriate constant value of the coning rate for the aerodynamic coefficients that are to be associated with an instantaneous flight condition is that formed from the instantaneous flight value of p_B/γ.

In summary, the components of the aerodynamic moment system in body axes take the form

$$\hat{C}_k(t) = \hat{C}_k(\infty\,;\ \hat{\alpha},\hat{\beta},p_B/\gamma)+ \frac{1}{\gamma}\frac{\dot{\hat{\alpha}}l}{V}[\hat{C}_{k_{q_B}}(\infty\,;\ \hat{\alpha},\hat{\beta},p_B/\gamma)+\gamma C_{k_{\dot{\hat{\alpha}}}}(\hat{\alpha},\hat{\beta},p_B/\gamma)] - \frac{1}{\gamma}\frac{\dot{\hat{\beta}}l}{V}[\hat{C}_{k_{r_B}}(\infty\,;\ \hat{\alpha},\hat{\beta},p_B/\gamma)-\gamma\hat{C}_{k_{\dot{\hat{\beta}}}}(\hat{\alpha},\hat{\beta},p_B/\gamma)],\ k = l,m,n \qquad (13)$$

In the body axis system, the three characteristic motions are coning, oscillations in pitch, and oscillations in yaw, all evaluated at constant values of angle of attack, angle of sideslip, and coning rate. The motions are illustrated schematically in Fig. 2.

Compatibility relations between the coefficients in the aerodynamic axis system in Eq. (4) and those in the body axis system in Eq. (11) can be obtained in the same way as described in Part I. [It is necessary first to equate arguments by expanding the terms in Eq. (11) to first order about $\hat{\omega}_1 = \omega_1$.] In particular, the interesting relation between the damping coefficients and the term $(C_{n_{\dot{\delta}}}-\gamma C_{n_{\dot{\psi}}})/\delta$ [Part I, Eq. (20)] can be shown to hold here as well at arbitrary ω_1 if $C_{n_{\dot{\phi}}}$ in Part I is replaced by $\partial C_n(\infty\,;\ \delta,\psi,\omega_1)/\partial(\omega_1 l/V)$. Further, it is easy to see that the simplified formulations represented by Eqs. (7) and (13) will revert to those of Part I upon expanding the terms in Eq. (7) to first order about $\dot{\lambda}/\gamma = 0$ and the terms in Eq. (13) to first order about $p_B/\gamma = 0$.

Spin radius

The simplified formulations in aerodynamic and body axes, Eqs. (7) and (13), have been obtained under the assumption that the body center of gravity make only small departures

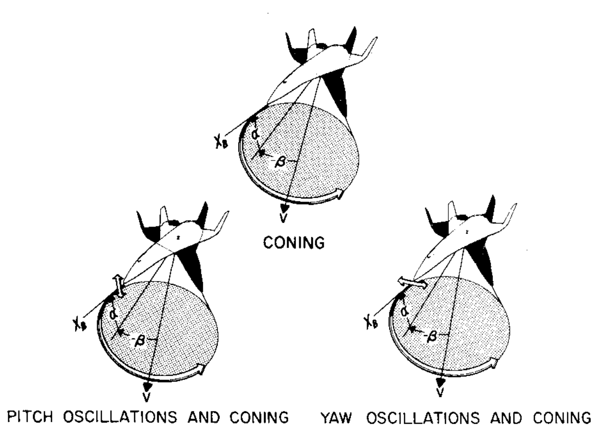

Fig. 2 Basic motions in body axis system.

from a rectilinear flight path. For motions involving classical spin, this would apparently restrict application of the results to those motions with essentially zero spin radius. It can be shown, however, that the results will apply as well to spin motions having constant spin radius. Motions having a constant spin radius can be characterized by the existence of a point other than the center of gravity about which the body rotates. This point, which lies on the body x_B axis, is itself in essentially rectilinear motion. The existence of such a point usually will guarantee fulfillment of the condition under which Eqs. (7) and (13) apply, namely, $q-\dot{\sigma} \approx 0, r-\varepsilon\dot{\lambda} \approx 0$. The principal restriction is that $qx/V, rx/V \ll 1$, where x is the distance along x_B between the center of gravity and the point in rectilinear motion. Variations in spin radius $x \sin \sigma$ also can be tolerated under the additional restriction $\dot{x}/V \ll 1$.

Underlying Assumptions

Although it is not necessary to do so, it has been assumed at the outset that flight velocity and atmospheric density remain constant during the motion and that the aircraft can be considered to be a rigid body. The assumption of constant velocity and density excludes from consideration the influence on motions of very large variations in trajectory properties such as might occur, e.g., during atmospheric re-entry. The rigid-body assumption is reflected in the functional forms, Eqs. (3) and (9), by the absence of structural variables. This omission rules out the possibility of treating the buffeting problem, which involves interactions between the elastic airframe and aerodynamic fluctuations.[3] As will be seen, however, the presence of fluctuations themselves can be acknowledged within the framework already established. The remaining assumptions are of two main classes: 1) fundamental assumptions associated with the use of functional analysis to develop a general integral form of the aerodynamic response; 2) simplifying assumptions associated with the reduction of the integral formulation to the more usable forms represented by Eqs. (7) and (13).

Fundamental assumptions

The principal assumption underlying the use of functional analysis concerns the existence and uniqueness of a nonlinear indicial response. An operational definition of the indicial pitching-moment response will be stated in a slightly more general way than was done in Ref. 4 in order to account for the presence of random fluctuations. For simplicity, the generalization will be described for the case of an independent variation of only one of the five motion variables in the aerodynamic axis system: resultant angle of attack $\sigma(t)$ about a rectilinear flight path, so that $q = \dot{\sigma}$. Extension to an arbitrary motion is straightforward.

As before, two maneuvers in $\sigma(\xi)$ are considered, beginning at $\xi = 0$, constrained at $\xi = \tau$, and differing only in the step imposed on the second maneuver at time τ (Fig. 3). For each maneuver, the pitching moment is measured at a time t

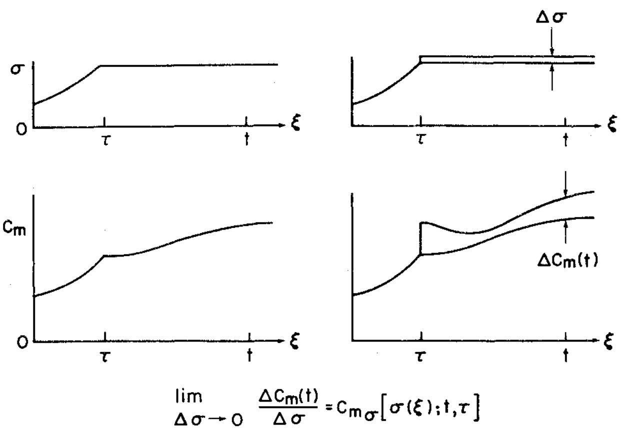

Fig. 3 Definition of nonlinear indicial response.

subsequent to τ. On the assumption of errorless maneuvers and repeatable measurements, the limit of the difference between measurements at time t, $\Delta C_m(t)$, divided by the magnitude of the step, $\Delta\sigma$, as $\Delta\sigma \to 0$, would be defined as the indicial response in pitching-moment coefficient per unit step change in σ. However, if $\sigma(\tau)$ is sufficiently large so that flow separation occurs in the course of a maneuver, then as a result of the ensuing fluctuations in the flow, any single measurement of the indicial response at time t will include a random component. This circumstance calls for repeating each maneuver and the corresponding measurement at time t many times and taking the arithmetic mean of the measurements. If the fluctuating part of the response is truly random its contribution to the measurement at time t should cancel in the mean and the resulting mean value should be representative of the deterministic part of the response. The principal assumption is that this is true for any time $t \geq \tau$ and that, as a result, for each maneuver a deterministic part of the response will exist that is continuous for all ξ in the interval $0 \leq \xi \leq t$. Similarly, the limit of the difference between mean values for the two maneuvers, $\Delta C_m(t)$, divided by the magnitude of the step, $\Delta\sigma$, as $\Delta\sigma \to 0$, also is assumed to exist and to be continuous for all ξ in the interval $\tau < \xi \leq t$ (the limit should be identically zero for $0 \leq \xi < \tau$ and may be discontinuous at $\xi = \tau$). With the understanding that the indicial response to each maneuver and at each time t is the result of an ensemble average of measurements, the indicial response in pitching-moment coefficient is defined as before:

$$\lim_{\Delta\sigma\to 0} (\Delta C_m(t)/\Delta\sigma) = C_{m_\sigma}[\sigma(\xi);\ t, \tau] \tag{14}$$

Strictly speaking, use of Eq. (14) to define the indicial response implies the exclusion of cases where the variation of steady-state pitching-moment response with σ becomes discontinuous either in its magnitude or slope at certain isolated values of σ, since $\lim_{\Delta\sigma\to 0}(\Delta C_m(\infty)/\Delta\sigma)$ will not exist at these points. Such cases are known to characterize certain types of stall behavior (cf, e.g., Ref. 5). Although these cases can be treated by an appropriate addition of jump conditions at the isolated points, in the interest of simplicity they will be excluded from further consideration here.

The use of ensemble averaging to define the indicial response should hold in all cases where the experimental evidence suggests that the indicial response does in fact have a unique deterministic part. The principal indicator of this would be if each of the repeated measurements of the steady-state response to a maneuver, $C_k[\sigma(\xi); \infty, \tau]$, $k = l, m, n$ yields a similar mean result, aside from small variations. The method will fail, however, under the following or related circumstances. Consider a long slender body of revolution which, after undergoing the maneuver $\sigma(\xi)$ over the interval $0 \leq \xi \leq \tau$, remains inclined at $\sigma(\tau)$ thereafter, and let the measuring time t be sufficiently removed from τ so that the flow over the body has reached steady state. Let $\sigma(\tau)$ be sufficiently large so that the steady separated flow pattern on the leeside of the body can be asymmetric, consisting of an odd number of leeside vortices. If there are no imperfections on the body that might bias the selection, it would appear that either a left-hand pattern of vortices or a right-hand pattern should be equally likely. A left-hand pattern would result in a side force in one direction, while a right-hand pattern would yield an equal side force in the opposite direction. Repeating the same maneuver many times, as is required in the ensemble averaging, should result in an equal number of left-hand and right-hand patterns so that the averaging would yield a zero steady-state side force, in contradiction of the result for any one maneuver. Thus, in general, if more than one steady-state response to the same maneuver is possible and if the steady-state response obtained in any one maneuver is probabilistic, then the assumptions underlying the method of defining an indicial response will fail. A later section will include a review of recent experimental evidence relating to the existence of multiple and probabilistic responses.

For those cases where the assumptions leading to the definition of a deterministic indicial response can be said to hold within each increment of the stepwise representation of an arbitrary motion $\sigma(t)$, the response $C_m(t)$ to the motion $\sigma(t)$ follows from a summation of incremental indicial responses over the time interval 0 to t:

$$C_m(t) = C_m(0) + \int_0^t C_{m_\sigma}[\sigma(\xi);\ t, \tau] \frac{d\sigma}{d\tau} d\tau \tag{15}$$

This is the general integral form for $C_m(t)$ to which a number of simplifying assumptions must be attached in order to reduce it to a more usable form.

Simplifying assumptions

In the form Eq. (15) the indicial response within the integral is itself a functional, depending in general on the whole past history of the motion $\sigma(\xi)$. This makes the further use of the form exceedingly difficult, since the history of the motion normally is not known in advance but rather is desired as the solution of the equations of motion. Thus, when the past history is unspecified, the functional also is unknown beforehand. Simplification of Eq. (15) hinges on replacing the functional by an appropriate function whose dependence on the past is denoted by a limited number of parameters rather than by a continuous function. If $\sigma(\xi)$ can be considered to be analytic in a neighborhood of $\xi = \tau$ (corresponding to the most recent past for an indicial response with origin at $\xi = \tau$), in principle its history can be reconstructed from a knowledge of all of the coefficients of its Taylor series expansion about $\xi = \tau$. Thus, since $\sigma(\xi)$ is equally represented by the coefficients of its expansion, the functional, with its dependence on $\sigma(\xi)$, can be replaced without approximation by a function with a dependence on all of the coefficients of the expansion of $\sigma(\xi)$ about $\xi = \tau$; i.e.,

$$C_{m_\sigma}[\sigma(\xi);\ t, \tau] = C_{m_\sigma}(t-\tau;\ \sigma(\tau), \dot{\sigma}(\tau), \ddot{\sigma}(\tau), \ldots) \tag{16}$$

The additional replacement of a dependence on *elapsed* time $t-\tau$ rather than on t and τ separately is justified within the assumptions already invoked of constant flight-path properties V and ρ. Now physical reasoning suggests that the indicial response should depend at most on only a limited interval of the most recent past. If it is assumed that this is true, then so far as the effect of the past on the indicial response is concerned, the form of the past motion just prior to the origin of the step might just as well have existed for all earlier times. Hence, at most only the first few coefficients of the expansion of $\sigma(\xi)$ need be retained to characterize correctly the most recent past, which is all the indicial response can be cognizant of. Retaining the first two coefficients, for example, implies matching the true past history $\sigma(\xi)$ in magnitude and slope at the origin of the step, thereby approximating the past by a linear function of time, $\sigma(\xi) \approx \sigma(\tau) - \dot{\sigma}(\tau)(\tau - \xi)$. With this approximation in force in Eq. (16), the integral form replacing Eq. (15) becomes

$$C_m(t) = C_m(0) + \int_0^t C_{m_\sigma}(t-\tau;\ \sigma(\tau), \dot{\sigma}(\tau)) \frac{d\sigma}{d\tau} d\tau \tag{17}$$

This form, while considerably more tractable than Eq. (15), is still sufficiently general to allow the treatment of motions involving hysteresis effects. Retaining a dependence on $\dot{\sigma}(\tau)$ allows assigning different indicial responses to a step at a single value of $\sigma(\tau)$, depending on the magnitude and sign of $\dot{\sigma}(\tau)$. It is possible, for example, to permit a distinction between indicial responses where σ was increasing or decreasing prior to the step. This will be valid for those cases where there is reason to assert that the particular response assigned to a step is a deterministic result of the past history rather than the probabilistic result of an interaction with a random fluctuation.

When it can be assumed that deterministic hysteresis effects are absent, and if the additional assumption is made that the motion is slowly varying, the dependence of the indicial response on $\dot{\sigma}(\tau)$ will not be significant. Omitting the dependence on $\dot{\sigma}(\tau)$ in Eq. (17) means that, so far as the indical response is concerned, the motion prior to the origin of a step is being approximated by the time-invariant motion $\sigma(\xi) \approx \sigma(\tau)$. The indicial response at any value of $t-\tau$, now dependent only on the magnitude of σ prior to the step, must not only be a continuous function of $\sigma(\tau)$, but henceforward also a *single-valued* function of $\sigma(\tau)$. With the dependence on $\dot{\sigma}(\tau)$ omitted, and under the consequent additional restriction of single-valuedness, Eq. (17) becomes the basic integral form underlying the simplified formulation. The remaining simplifications are consistent with the assumption of a slowly varying motion. Putting in evidence the steady-state value of the indicial response by the substitution

$$C_{m_\sigma}(t-\tau;\sigma(\tau)) = C_{m_\sigma}(\infty;\sigma(\tau)) - F(t-\tau;\sigma(\tau)) \qquad (18)$$

enables the integration of the steady-state term in Eq. (17), yielding

$$C_m(t) = C_m(\infty;\sigma(t)) - \int_0^t F(t-\tau;\sigma(\tau))\frac{d\sigma}{d\tau}d\tau \qquad (19)$$

The deficiency function $F(t-\tau;\sigma(t))$ is essentially zero for all elapsed times $t-\tau$ larger than a relatively small value t_a. Since the motion $\sigma(\tau)$ is slowly varying, to a first order in frequency it suffices to retain only the first term of an expansion of $\dot{\sigma}(\tau)$ around $\tau = t$. For $t \gg t_a$, Eq. (19) becomes

$$C_m(t) = C_m(\infty;\sigma(t)) - \frac{l}{V}\dot{\sigma}(t)\left(\frac{V}{l}\int_0^{t_a} F(u;\sigma(t))\,du\right) \qquad (20)$$

where, in $F(u;\sigma(t))$, the approximation $\sigma(t-u) \approx \sigma(t)$ has been imposed on the basis of $t \gg t_a$. After the integral term in parentheses is identified with $-C_{m_{\dot{\sigma}}}(\sigma(t))$, Eq. (20) is recognized as the simplified formulation in the aerodynamic axis system with $\psi, \omega_1, \omega_3 = 0, \omega_2 = q = \dot{\sigma}$ [cf. Eq. (7)].

Discussion

Multiple and Probabilistic Responses

It has been noted that the assumptions underlying the present analysis will fail if multiple steady-state responses to a given maneuver exist and if the steady-state response obtained in any one maneuver is probabilistic. Failure of the present method aside, there is reason to be concerned over the possibility that probabilistic responses may occur in the maneuvers of modern high-speed aircraft at high angles of attack. One case has been noted, that of the slender body of revolution at high angle of attack, where the phenomenon may exist because of the presence and equal likelihood of left-hand and right-hand asymmetric leeside vortex patterns. Conditions similar to those of the slender body of revolution would appear to be created by the long slender forebodies of modern aircraft. Thus, the question whether these aircraft also are susceptible to multiple and probabilistic responses deserves careful attention from experimenters. The following is a brief summary of recent experimental evidence in this regard.

In Ref. 6, perhaps the most extensive study to date of the vortex patterns over inclined slender cylindrical bodies, it was found that the asymmetric vortex patterns obtained at moderate angles of attack were generally steady and repeatable (i.e., deterministic). The mechanism governing the selection of one pattern rather than its reverse was attributed to a minute misalignment of the nose section. It would seem likely, however, that a successive reduction in the magnitude of the misalignment might eventually make the selection probabilistic. In Ref. 7, it is reported that repeated tests under similar wind-tunnel conditions on a model of a fighter-type aircraft at high angles of attack and zero sideslip angle yielded widely different steady side-force results from one test to the next. Here, minute uncontrollable asymmetries near the nose were cited as the source of the randomness, so that a given response would have to be termed probabilistic.

Although the experimental evidence is not conclusive, it does suggest that the presence of uncontrollable imperfections near the nose of a slender aircraft might cause the occurrence of probabilistic aerodynamic responses during certain high-angle-of-attack maneuvers. Since their occurrence would make the maneuvers predictable only in a statistical sense, it might be preferable and, as also suggested by the experimental evidence, relatively easy to ensure a deterministic response by the deliberate installation of a controlling imperfection.

Deterministic Hysteresis Effects

In the absence of probabilistic aerodynamic responses, the use of the simplified formulation [Eq. (7) or Eq. (13)] to describe the aerodynamic moment is still precluded if the indicial responses are significantly affected by the past motion. However, such cases fall within the scope of the integral form exemplified by Eq. (17) and, in principle, could be treated by a version of Eq. (17) suitably generalized to include the remaining variables.

For the special case exemplified by Eq. (17), it will be noted that each of the two maneuvers required to define an indicial response (cf. Fig. 3) is essentially the same maneuver that has been used in the past to study the over-shoot in lift following a rapid pitch-up. The wind-tunnel experiments devised for this purpose (cf. in particular, Ref. 8) would appear to be the most appropriate ones to determine whether the indicial response is in fact significantly affected by the past motion. To cover the most important effect of the past motion, however, these experiments should be extended to include both pitch-up and pitch-down maneuvers to a given stationary angle of attack. This procedure would allow determining whether the *steady-state* value of the response following a maneuver is independent of the rate, $\dot{\sigma}$, of the pitch-up or pitch-down maneuver, or whether it can be multi-valued, dependent on the magnitude or, more importantly, on the sign of $\dot{\sigma}$. The latter event is known to occur for unswept wings of large aspect ratio in the stall regime. There is some experimental evidence, although at low Reynolds numbers,[9] that similar events occur for swept wings of low aspect ratio in the phenomenon of "vortex breakdown" to which the leading-edge vortices on these wings are known to be susceptible at high angles of attack.[10] Additional tests at higher Reynolds numbers will be needed to determine whether these hysteresis effects are important enough to require adopting the necessary generalization of Eq. (17).

Simplified Formulation

When both probabilistic aerodynamic responses and deterministic hysteresis effects are absent, use of the simplified formulation is then restricted primarily by the principal remaining assumption, namely, that the coning rate is the only component of the angular velocity vector that can significantly affect the aerodynamic coefficients. Cases can be envisaged where this will not be true. The aerodynamics of aircraft in purely rolling maneuvers, for example, clearly will be more significantly influenced by the roll rate p_B than by the coning rate. In general, it is the initial choice of a basic steady motion, typical of a class of motions, that will determine which of the components of the angular velocity vector should be the most influential. For cases where there is a point on the longitudinal axis of the aircraft that does not depart significantly from a rectilinear path, it is reasonable to choose coning motion as the basic steady motion. The simplified formulation presented here should be applicable in these cases.

Wind-Tunnel Experiments with Simplified Formulation

Requirements for experiments

According to the analysis in either the aerodynamic or the body axis system, the moment contributions from three characteristic motions are required to specify the moment system for arbitrary motions: a coning motion and two oscillatory motions in the presence of coning (cf. Figs. 1, 2). Experiments designed to reproduce the motions in the wind-tunnel require the same coning apparatus and the same types of oscillatory devices already described in Part I. The significant additional requirement that each of the oscillatory experiments be carried out in the presence of coning means, however, that now the oscillatory devices must be incorporated in the coning apparatus. These obviously difficult experiments, involving oscillatory and coning motions in combination, are required only where the moment contribution due to steady coning shows a significant *nonlinear* dependence on coning rate. Otherwise, the experiments may be conducted separately as described in Part I. Useful surveys of the needs and capabilities for carrying out experiments in the wind-tunnel involving oscillatory, rotary, and combined motions at high angles of attack are available in Refs. 11 and 12.

Scale effects

The asymmetric separated flow patterns that can exist on the leeside of long slender bodies at high angles of attack appear to be sensitive to Reynolds-number variation. As shown in Ref. 6, at fixed incidence and Mach number an increase in the Reynolds number beyond a certain value caused the steady asymmetric vortex pattern to become unsteady. A further incremental increase in the Reynolds number resulted in a steady pattern again, somewhat shifted from the previous steady pattern. Since this behavior is suggestive of successive regimes of stable and unstable flows, it is reasonable to presume that further increases in the Reynolds number would result in additional successions of stable and unstable regimes. Thus, the vortex patterns observed in wind-tunnel tests of slender aircraft may not correspond to those that would be observed at the very much higher Reynolds numbers typical of full-scale flight conditions. The existence of asymmetric vortex flows sensitive to Reynolds-number variations presents another aspect of the problem of wind-tunnel scale effects that may not yield to the ameliorative procedures used in the past in connection with boundary-layer flows. It would appear advisable to test at least one configuration at one flight condition (V, σ) over as wide a Reynolds-number range as possible to determine how and to what extent the vortex patterns respond as the Reynolds number is increased from wind-tunnel to full-scale values. Additionally, since the influence of coning rate on the vortex patterns will be similar in certain respects to a Reynolds-number effect, it may be necessary to include coning rate as a variable in such an investigation.

Suggested experiment

In order to verify whether the formulation presented here adequately describes the moment system during high-angle-of-attack maneuvers, including those in the stall and post-stall regimes, a procedure originally recommended in Ref. 2 is endorsed here. It is suggested that the three wind-tunnel experiments required by the formulation be carried out on a model of a slender fighter-type aircraft. Free-flight spin tests of a geometrically similar model should be carried out in a spin-tunnel at the same flight speed and Reynolds number. Carrying out all the experiments at one Reynolds number would eliminate the anticipated problem of scale effects that might otherwise cloud the issue, so far as verification of the formulation is concerned. The adequacy of the formulation would be verified if the motions predicted using measured coefficients of an aerodynamic moment system having the form specified here agreed with those recorded in the spin-tunnel experiments.

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